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Multi-soliton and multi-cuspon solutions of a Camassa–Holm hierarchy and their interactions

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Abstract

In this paper, we study an integrable Camassa–Holm hierarchy whose high-frequency limit is the Camassa–Holm equation. By a method associated with the Darboux transform, we construct the explicit multi-soliton and multi-cuspon solutions. Then, we study in detail the interactions of soliton–cuspon and soliton–soliton. Further, an interesting phenomenon is found: the soliton with smaller amplitude can travel faster than the one with larger amplitude when they interact. At last, we investigate in detail the head-on collision of one soliton and one cuspon.

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1. Introduction

The Camassa–Holm (CH) equation

$$u_t + 2\omega u_x + 3uu_x - u_{xxt} - 2u_x u_{xx} - uu_{xxx} = 0 \tag{1}$$

can be determined by the isospectral problem

$$\partial_{x,x}\psi = \left(\frac{1}{4} + \lambda q\right)\psi, \tag{2}$$

$$\partial_t\psi = \left(\frac{1}{2\lambda} - u\right)\partial_x\psi + \frac{u_x}{2}\psi, \tag{3}$$

where $q = u - u_{xx} + \omega$ and ω is a real constant. Equation (1) was proposed in Camassa and Holm [1] and Camassa *et al* [2] as a model equation for unidirectional nonlinear dispersive waves in shallow water. It was also found to be a model for nonlinear dispersive waves in hyperelastic rods (see Dai [3]). This equation has attracted considerable attention due to its complete integrability for all values of ω and other mathematical properties, e.g., when $\omega = 0$, Camassa and Holm [1] found the peakon solution of the form

$$\rho(x,t) = V\,\mathrm{e}^{-|x-Vt|}, \qquad V \neq 0. \tag{4}$$

An important feature of the CH equation is that the above solution has the characteristic that the wave crest has a corner form, which is shared by waves of the greatest height in water (cf [4–7]). Also, it has been proven that the peakons and all smooth solitary waves for this equation are orbitally stable (see [8–11]). This equation also admits many other interesting traveling waves (see [12–13]), including a number of weak solutions (cf [14–15]). However, because of the complexity of the CH equation, it is considerably difficult to find its other solutions, especially, the N-soliton solution, N-cuspon solution, etc. Recently, a large number of works have been devoted to study these solutions of the CH equation.

In Ferreira *et al* [16], they investigated the interaction of a soliton and a cuspon with the help of numerical methods for the case of $\omega = 2$, as no explicit analytical expressions were available at the time. Nevertheless, the authors managed to obtain some analytical results for phase shifts after the interaction. Johnson [17] implemented Constantin's scattering approach (see [18]) and only obtained the analytic two-soliton and three-soliton solutions. More recently, Parker ([19–21]) managed to obtain the associated bilinear form of the CH equation and analyzed the N-soliton solution with the additional parameter ($N \geqslant 2$). Matsuno [22] acquired a parametric representation for the N-soliton solution of (1) by introducing an appropriate coordinate transform. Constantin *et al* [23] have given a formal expression for the N-soliton solution by using the inverse scattering transform(IST). Further, in [24], Constantin *et al* have extended the IST in [23] as a generalized Fourier transform to a CH hierarchy and shown the fundamental characteristics of all the equations of the whole CH hierarchy. In Li and Zhang [25] and Li [26], a different approach associated with the Darboux transform was introduced to construct the explicit expressions for the multi-soliton solution. Dai and Li [27] have developed the conclusion of [25–26] to soliton and cuspon solutions of (1). Further, using the newly available explicit multi-soliton and multi-cuspon solutions of the CH equation, the authors investigated the two-wave interactions and revealed some interesting phenomena that were not found in the numerical work of Ferreira *et al* [16]. Recently, the stability for trains of solitary waves was studied by Dika and Molinet [28].

In this paper, we consider a complete Camassa–Holm hierarchy (CHH). From the over-determined isospectral problem

$$\partial_{x,x}\psi = \left(\frac{1}{4} + \lambda q\right)\psi, \tag{5}$$

$$\partial_t\psi = \frac{u}{\frac{1}{\lambda} - \epsilon}\partial_x\psi - \frac{\partial_x u}{2\left(\frac{1}{\lambda} - \epsilon\right)}\psi, \tag{6}$$

which is first introduced in [29], we immediately obtain the CHH

$$q_t = \tfrac{1}{2}(u_x - u_{xxx}), \qquad -\infty < x < \infty, \tag{7}$$

$$\epsilon q_t + u q_x + 2 q u_x = 0, \tag{8}$$

where ϵ is the hierarchy parameter and λ is the spectral parameter. Dai and Pavlov [29] showed that its high-frequency limit is the CH equation and its low-frequency limit is the

Hunter–Saxton equation, which is a model for the motion of a nematic liquid crystal (see [30]). The global solutions of this equation were studied by Bressan and Constantin [31]. The main purpose of this paper is to construct the N-soliton and N-cuspon solutions of the above hierarchy and examine the interactions of soliton–cuspon and soliton–soliton, with the aid of explicit solutions. An interesting phenomenon is found: the soliton with smaller amplitude can travel faster than the one with larger amplitude when they interact.

2. The solutions of the CHH

We introduce the transformation

$$\phi(y) = q^{\frac{1}{4}}\psi, \tag{9}$$

$$\partial_x y = \sqrt{q}. \tag{10}$$

This transforms (5) into

$$-\partial_{y,y}\phi + Q(y)\phi = \mu\phi, \qquad \mu = -\frac{1}{4\omega} - \lambda, \tag{11}$$

$$Q(y) = \frac{1}{4q} + \frac{\partial_{y,y}q}{4q} - \frac{3(\partial_y q)^2}{16q^2} - \frac{1}{4\omega}. \tag{12}$$

In Li and Zhang [25], it has been shown that (11) and (12) have the following solutions:

Denote the two fundamental solutions of (11) with zero potential as

$$\Phi_i = \begin{cases} \cosh k_i y, & i \text{ is odd,} \\ \sinh k_i y, & i \text{ is even,} \end{cases} \tag{13}$$

where $\lambda = k_i^2 - \frac{1}{4\omega}$.

Then we have

$$q = g_1^2 g_2^2, \tag{14}$$

where

$$g_1 = \frac{f_1}{\sqrt{\Delta_1\Delta_2}}, \qquad g_2 = \frac{f_2}{\sqrt{\Delta_1\Delta_2}}, \tag{15}$$

$$\Delta_1 = \prod_{i=1}^{n}\left(\frac{1}{\sqrt{\omega}} - k_i\right), \qquad \Delta_2 = \prod_{i=1}^{n}\left(\frac{-1}{\sqrt{\omega}} - k_i\right), \tag{16}$$

$$f_1 = \frac{W\left(\Phi_1, \Phi_2, \ldots, \Phi_n, \mathrm{e}^{\frac{y}{2\sqrt{\omega}}}\right)}{W(\Phi_1, \Phi_2, \ldots, \Phi_n)}, \qquad f_2 = \frac{W\left(\Phi_1, \Phi_2, \ldots, \Phi_n, \mathrm{e}^{\frac{-y}{2\sqrt{\omega}}}\right)}{W(\Phi_1, \Phi_2, \ldots, \Phi_n)}. \tag{17}$$

Here, $W(\Phi_1, \Phi_2, \ldots, \Phi_n)$ is the Wronskian determinant. The parameter y is related to x through

$$x = \log\left(\sqrt{\frac{f_1^2}{f_2^2}}\right). \tag{18}$$

Further,

$$Q(y) = -2\partial_{y,y}\log[W(\Phi_1, \Phi_2, \ldots, \Phi_n)]. \tag{19}$$

Also, the solution has the behavior $q \to \omega$ as $y \to \pm\infty$.

Now we have to introduce the variable t from the t part of the Lax pair. We note that $q = \omega, u = 1$ is a solution of the CHH. In this case, we have $\partial_t\psi = \omega^{-\frac{1}{4}}\partial_t\phi = \frac{1}{\frac{1}{\lambda}-\epsilon}\partial_x\psi = \frac{\omega^{\frac{1}{4}}}{\frac{1}{\lambda}-\epsilon}\partial_y\phi$. It yields that $\partial_t\phi = \frac{\sqrt{\omega}}{\frac{1}{\lambda}-\epsilon}\partial_y\phi$. It means that in the above solution we need to change $k_i y$ to

$$\xi_i = k_i\left(y + \frac{\sqrt{\omega}t}{\frac{1}{k_i^2-\frac{1}{4\omega}}-\epsilon}\right). \tag{20}$$

In order to find u, we rewrite (8) as

$$\begin{aligned}\epsilon\partial_t\sqrt{q} + \partial_x(\sqrt{q}u) &= \epsilon\frac{q^{\frac{-1}{2}}\partial_t q}{2} + \frac{q^{\frac{-1}{2}}u\partial_x q}{2} + q^{\frac{1}{2}}\partial_x u \\ &= \frac{q^{-\frac{1}{2}}(\epsilon\partial_t q + u\partial_x q + 2q\partial_x u)}{2} = 0.\end{aligned}$$

By using $\sqrt{q} = \partial_x y$ for the above equation, it yields that $0 = \epsilon\partial_t(\partial_x y) + \partial_x(u\partial_x y) = \epsilon\partial_x(\partial_t y) + \partial_x(u\partial_x y)$. As a result, we have

$$u = -\epsilon\frac{\partial_t y}{\partial_x y} + \frac{c}{\partial_x y}, \tag{21}$$

where c is a constant.

Let $F(x, y, t) = x - \log\left(\sqrt{\frac{f_1^2}{f_2^2}}\right) = 0$, then $\mathrm{d}F = \partial_x F\,\mathrm{d}x + \partial_y F\,\mathrm{d}y + \partial_t F\,\mathrm{d}t = 0$, and $\frac{\partial_t y}{\partial_x y} = \partial_t F$, $\partial_x y = -\frac{1}{\partial_y F}$, $\partial_t y = -\frac{\partial_t F}{\partial_y F}$. So, we have

$$u = -\epsilon\partial_t F + \frac{c}{\partial_x y} = \epsilon\partial_t\left(\log\frac{f_1}{f_2}\right) + \frac{c}{\partial_x y} = \epsilon\left(\frac{\partial_t f_1}{f_1} - \frac{\partial_t f_2}{f_2}\right) + \frac{c}{\sqrt{q}}. \tag{22}$$

The constant c can be determined by substituting (22) and (14) into (8).

We also note that the following symmetry: if $[u(x, t, \epsilon), q(x, t, \epsilon)]$ is a pair of solutions of the CHH, $[-u(x, t, -\epsilon), -q(x, t, -\epsilon)]$ and $[-u(x, -t, \epsilon), q(x, -t, \epsilon)]$ are other two pairs of solutions. Therefore, once a pair of solutions is found, other two pairs of solutions can be deduced by using the above symmetry.

3. Single (ϵ, ω)-soliton solutions and single (ϵ, ω)-cuspon solutions

When two solitons or two cuspons are well separated, each of them can be regarded as a single soliton or a single cuspon. So, in this section we first represent the solutions of (ϵ, ω)-soliton and (ϵ, ω)-cuspon.

3.1. (ϵ, ω)-soliton solutions

We take $n = 1$ in (14)–(17) and

$$\Phi_1 = \cosh\xi_1, \qquad \xi_1 = k_1\left(y + \frac{\sqrt{\omega}t}{\frac{1}{k_1^2-\frac{1}{4\omega}}-\epsilon}\right), \qquad \omega > 0.$$

As a result, we obtain

$$f_1 = \frac{\exp\left(\frac{y}{2\sqrt{\omega}}\right)(1 - 2k_1\sqrt{\omega}\tanh\xi_1)}{2\sqrt{\omega}},$$

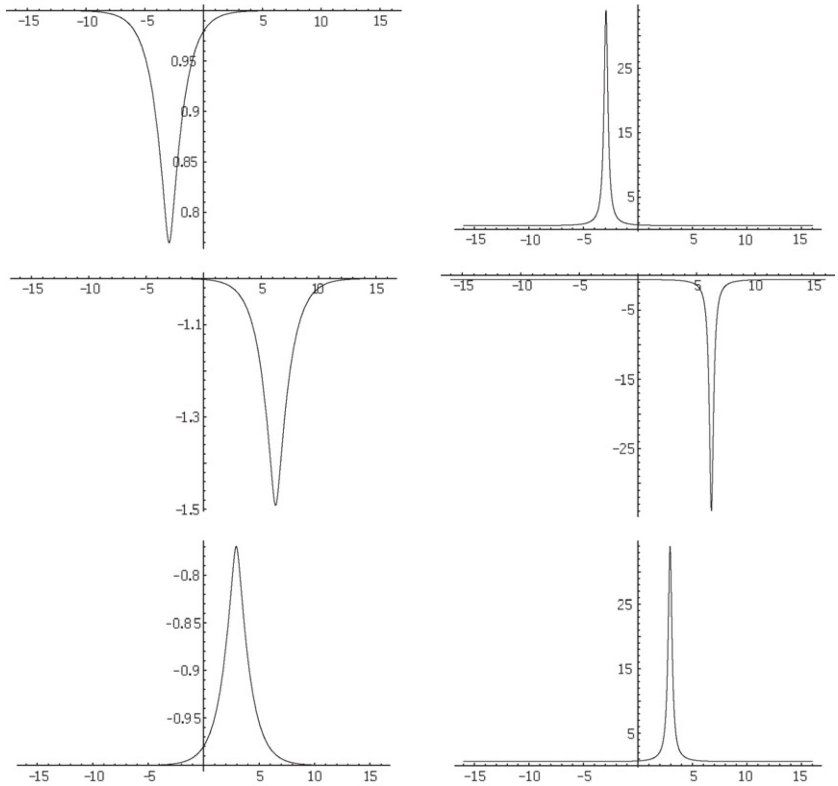


Figure 1. The three different soliton solution curves of $[u, q]$ for $k_1 = 0.6, \omega = 0.602, \epsilon = 50, t = -200$.

$$f_2 = \frac{\exp\left(\frac{-y}{2\sqrt{\omega}}\right)(1 + 2k_1\sqrt{\omega}\tanh\xi_1)}{2\sqrt{\omega}},$$

$$u(y, t) = -\frac{\left(4k_1^2\omega - 1\right)\left(\epsilon + 4\omega - 4\epsilon k_1^2\omega + \frac{4\epsilon k_1^2\omega}{\cosh^2\xi_1}\right)}{\left(-4\omega + \epsilon\left(4k_1^2\omega - 1\right)\right)\left(4k_1^2\omega\tanh^2\xi_1 - 1\right)},$$

$$\sqrt{q} = \frac{\sqrt{\omega}\left(4k_1^2\omega\tanh^2\xi_1 - 1\right)}{4k_1^2\omega - 1},$$

$$x = \log\left(\sqrt{\frac{(2k_1\sqrt{\omega}\tanh\xi_1 - 1)^2\,\mathrm{e}^{\frac{2y}{\sqrt{\omega}}}}{(2k_1\sqrt{\omega}\tanh\xi_1 + 1)^2}}\right).$$

When $2k_1\sqrt{\omega} < 1$, y and x are one-to-one correspondent. The functions $u, q, \partial_x u$ and $\partial_x q$ are continuous in $(-\infty, \infty)$, and the maximum value u is at $t = y = 0$ and has the value

$$u(0, 0) =: hs = \frac{1 + \frac{\epsilon}{4\omega}}{\frac{\epsilon}{4\omega} - \frac{1}{4k_1^2\omega - 1}}. \tag{23}$$

From (23), we arrive at

$$hs \to 1, \qquad \frac{\epsilon}{4\omega} \to \pm\infty.$$

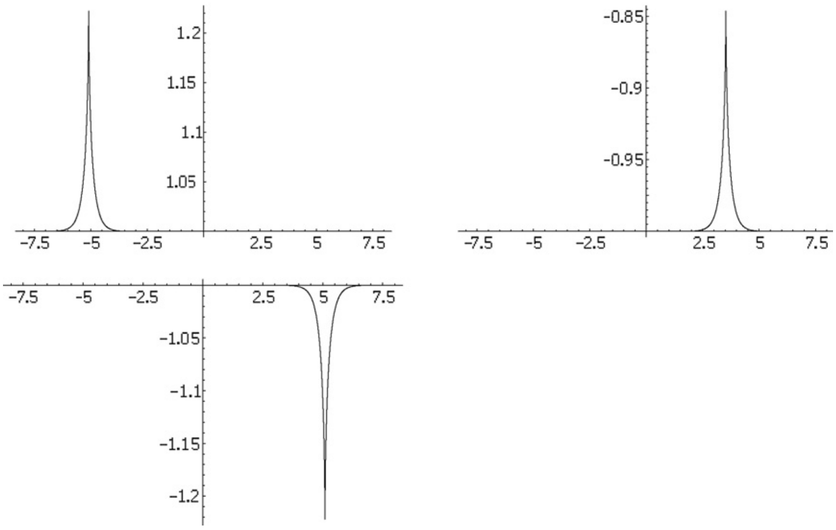


Figure 2. The three different cuspon solution curves of u for $k_1 = 0.698, \omega = 8.8, \epsilon = 12, t = -50$.

Figure 1 describes three different solution curves of $[u, q]$, respectively, for

$[u(x, t, \epsilon), q(x, t, \epsilon)], [-u(x, t, -\epsilon), -q(x, t, -\epsilon)], [-u(x, -t, \epsilon), q(x, -t, \epsilon)]$.

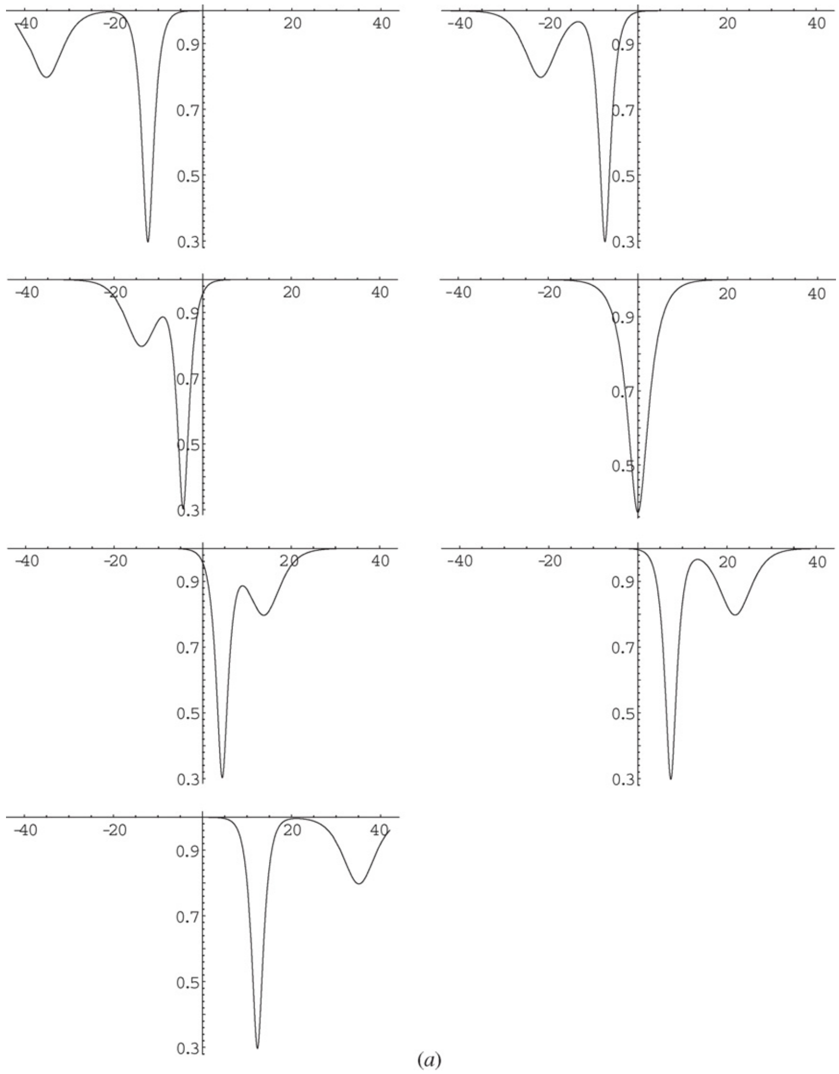
3.2. (ϵ, ω)-cuspon solutions

We take $n = 1$ in (14)–(17) and

$$\Phi_1 = \sinh \xi_1, \qquad \xi_1 = k_1 \left(y + \frac{\sqrt{\omega} t}{\frac{1}{k_1^2 - \frac{1}{4\omega}} - \epsilon} \right), \qquad \omega > 0.$$

Then, we have

$$f_1 = \frac{\exp\left(\frac{y}{2\sqrt{\omega}}\right)(1 - 2k_1\sqrt{\omega}\coth \xi_1)}{2\sqrt{\omega}},$$
$$f_2 = \frac{\exp\left(\frac{-y}{2\sqrt{\omega}}\right)(1 + 2k_1\sqrt{\omega}\coth \xi_1)}{2\sqrt{\omega}},$$
$$u(y, t) = \frac{\left(4k_1^2\omega - 1\right)\left(4\epsilon k_1^2\omega + \left(-4\omega - \epsilon(4k_1^2\omega - 1\right)\sinh^2 \xi_1)\right)}{\left(-4\omega + \epsilon\left(4k_1^2\omega - 1\right)\right)\left(4k_1^2\omega \cosh^2 \xi_1 - \sinh^2 \xi_1\right)},$$
$$\sqrt{q} = \frac{\sqrt{\omega}\left(4k_1^2\omega \coth^2 \xi_1 - 1\right)}{4k_1^2\omega - 1},$$
$$x = \log\left(\sqrt{\frac{(2k_1\sqrt{\omega}\coth \xi_1 - 1)^2 \,\mathrm{e}^{\frac{2y}{\sqrt{\omega}}}}{(2k_1\sqrt{\omega}\coth \xi_1 + 1)^2}}\right).$$



(*a*)

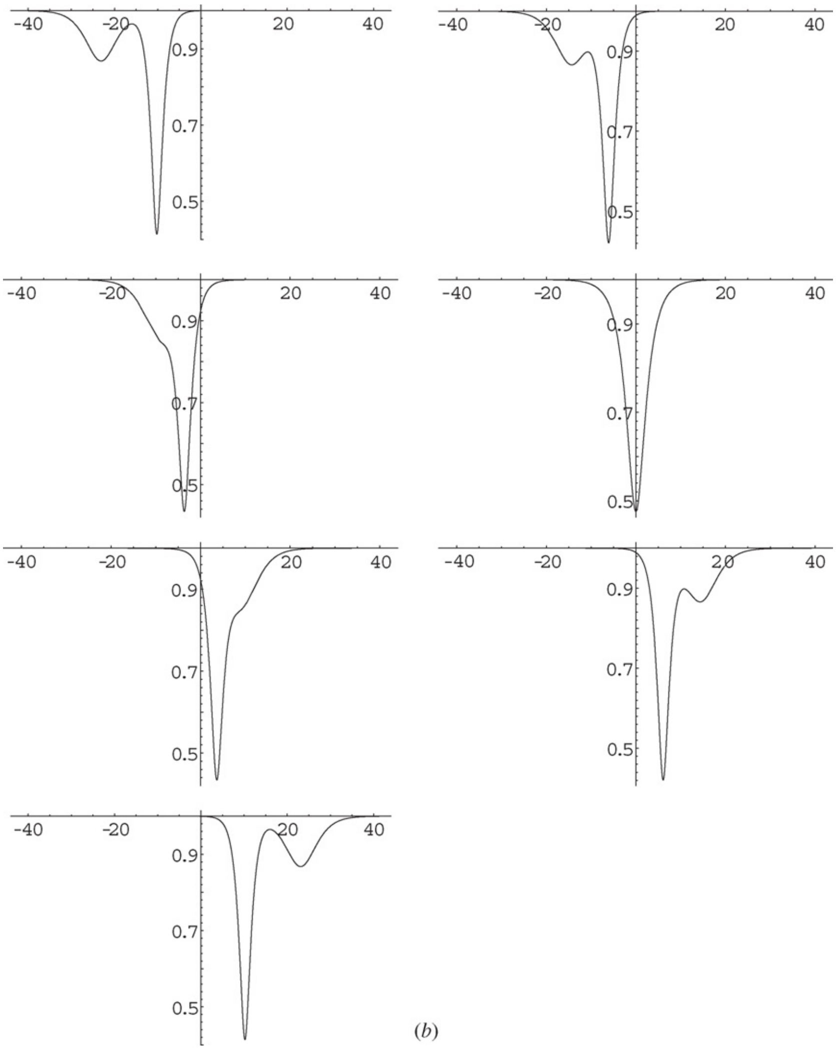
Figure 3. (*a*) The interaction of two (ϵ, ω) solitons of u for $k_1 = 0.2, k_2 = 0.4, \omega = 1, \epsilon = -1$. From top to bottom, the time $t = (-130, -80), (-50, 0), (50, 80), 130$. (*b*) The interaction of two (ϵ, ω)-solitons of u for $k_1 = 0.2, k_2 = 0.4, \omega = 1, \epsilon = 1$. From top to bottom, the time $t = (-130, -80), (-50, 0), (50, 80), 130$.

When $2k_1\sqrt{\omega} > 1$, y and x are one-to-one correspondent. The solution u is continuous in $(-\infty, \infty)$, but $\partial_x u$ and q have singularities, and the maximum value u is at $t = y = 0$ and has the value

$$u(0,0) =: hc = \frac{\epsilon}{\epsilon - \frac{4\omega}{4k_1^2\omega - 1}}. \tag{24}$$

From (24), we also arrive at

$$hc \to 1, \qquad \epsilon \to \pm\infty.$$



(*b*)

Figure 3. (Continued.)

In figure 2, we describe three different cuspon solution curves of u, respectively, for

$$u(x, t, \epsilon), -u(x, t, -\epsilon), -u(x, -t, \epsilon).$$

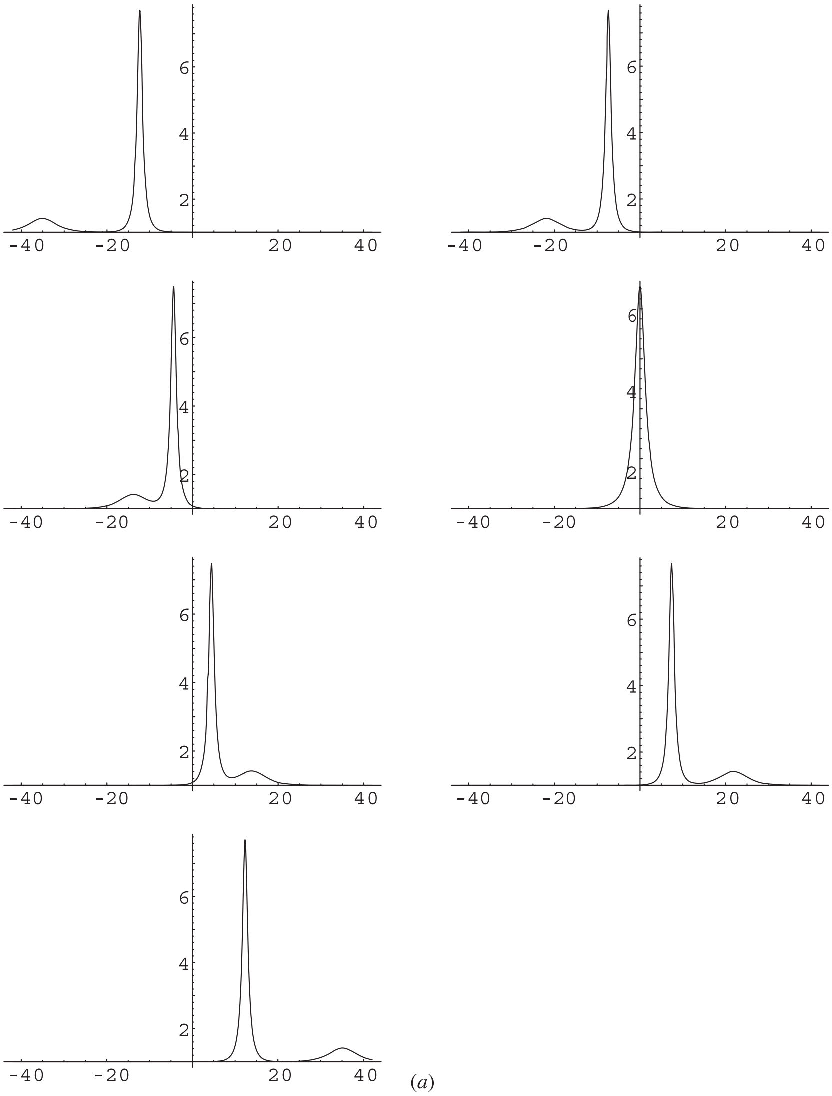
However the solution curves of q cannot be plotted well for its singularities.

4. The interaction processes

4.1. The interaction of two (ϵ, ω)*-solitons*

We take $n = 2$ in (14)–(17) and set

$$W(\Phi_1, \Phi_2) = W(\cosh \xi_1, \sinh \xi_2),$$



(*a*)

Figure 4. (*a*) The interaction of two (ϵ, ω) solitons of q for $k_1 = 0.2, k_2 = 0.4, \omega = 1, \epsilon = -1$. From top to bottom, the time $t = (-130, -80), (-50, 0), (50, 80), 130$. (*b*) The interaction of two (ϵ, ω) solitons of q for $k_1 = 0.2, k_2 = 0.4, \omega = 1, \epsilon = 1$. From top to bottom, the time $t = (-130, -80), (-50, 0), (50, 80), 130$.

$$f_1 = \frac{W\big(\cosh\xi_1, \sinh\xi_2, \exp\big(\frac{y}{2\sqrt{\omega}}\big)\big)}{W(\cosh\xi_1, \sinh\xi_2)},$$

$$f_2 = \frac{W\big(\cosh\xi_1, \sinh\xi_2, \exp\big(\frac{-y}{2\sqrt{\omega}}\big)\big)}{W(\cosh\xi_1, \sinh\xi_2)},$$

$$u = \epsilon\partial_t \ln\left(\frac{f_1}{f_2}\right) + \sqrt{\frac{\omega}{q}}, \qquad x = \log\left(\sqrt{\frac{f_1^2}{f_2^2}}\right),$$

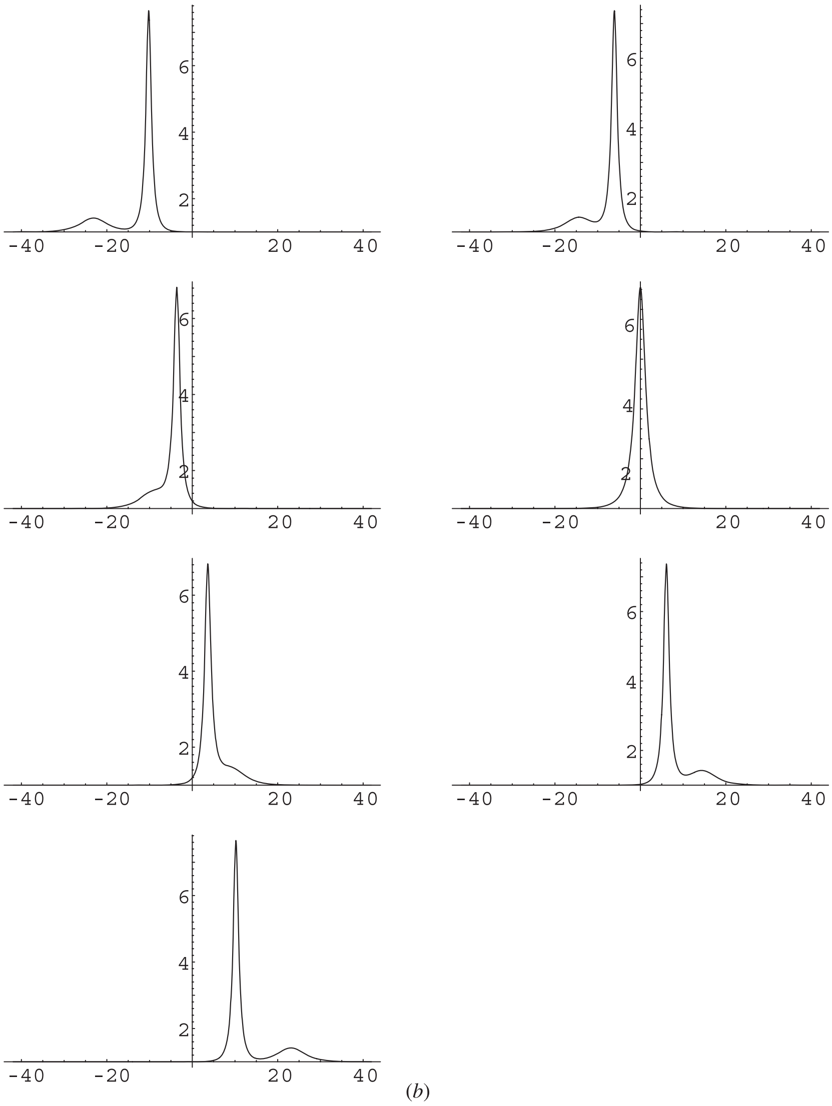


Figure 4. (Continued.)

$$\xi_1 = k_1\left(y + \frac{\sqrt{\omega}t}{\frac{1}{k_1^2-\frac{1}{4\omega}} - \epsilon}\right), \qquad \xi_2 = k_2\left(y + \frac{\sqrt{\omega}t}{\frac{1}{k_2^2-\frac{1}{4\omega}} - \epsilon}\right), \qquad \omega > 0.$$

The corresponding solution is a two-soliton solution.

In figure 3(*a*) ($\epsilon = -1$) and in figure 3(*b*) ($\epsilon = 1$), we have described the interaction processes of two (ϵ, ω)-solitons for seven different times of u, respectively. It can be seen that the interaction processes and the shapes of two (ϵ, ω)-solitons in figures 3(*a*) and (*b*) are similar. That is, the sign of ϵ (other parameters are identical) has little influence on the characters of the two-soliton solution of u. The interaction is very similar to that of two solitons of the KdV equation.

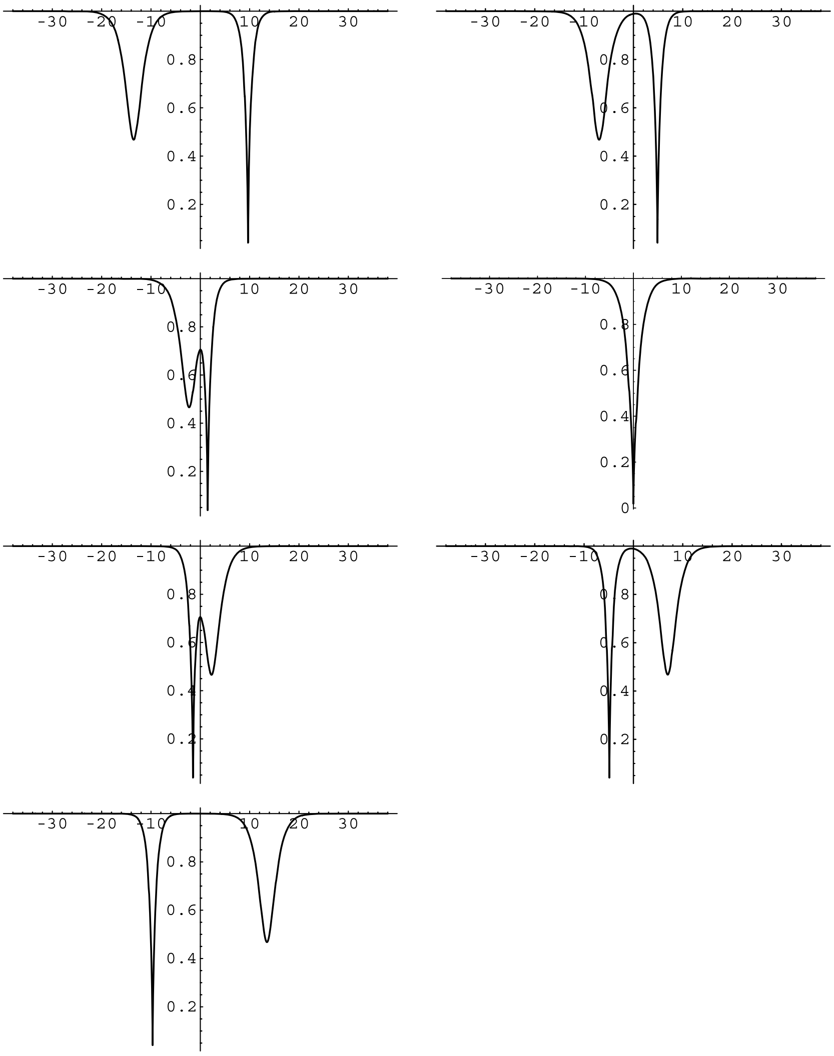


Figure 5. The interaction of one (ϵ, ω)-soliton and one (ϵ, ω)-cuspon of u for $k_1 = 0.3, k_2 = 0.5, \omega = 1.4, \epsilon = -0.6$. From top to bottom, the time $t = (-150, -80), (-30, 0), (30, 80), 150$.

Figure 4(a) ($\epsilon = -1$) and figure 4(b) ($\epsilon = 1$) plot the interaction processes of two (ϵ, ω)-solitons for seven different times of q, respectively. From the graphs, it is easy to see that the sign of ϵ (other parameters are identical) has little influence on the characters of the two-soliton solution of q. Initially, at $t = -130$, they are well separated and the soliton with larger amplitude is located at the right. At $t = 0$, the soliton with smaller amplitude catches up with another soliton with larger amplitude and they merge into one. Afterward, they gradually separate as time increases and eventually at $t = 130$, the two solitons recover their original shapes apart from some phase shifts. An interesting phenomenon is that the soliton with smaller amplitude travels faster than the one with larger amplitude. This is opposite to the usual case that a soliton with larger amplitude travels faster.

4.2. The interaction of a (ϵ, ω)-soliton and a (ϵ, ω)-cuspon.

We take $n = 2$ in (14)–(17) and set

$$W(\Phi_1, \Phi_2) = W(\cosh\xi_1, \cosh\xi_2),$$

$$f_1 = \frac{W\left(\cosh\xi_1, \cosh\xi_2, \exp\left(\frac{y}{2\sqrt{\omega}}\right)\right)}{W(\cosh\xi_1, \cosh\xi_2)},$$

$$f_2 = \frac{W\left(\cosh\xi_1, \cosh\xi_2, \exp\left(\frac{-y}{2\sqrt{\omega}}\right)\right)}{W(\cosh\xi_1, \cosh\xi_2)},$$

$$u = \epsilon\partial_t \ln\left(\frac{f_1}{f_2}\right) + \sqrt{\frac{\omega}{q}}, \qquad x = \log\left(\sqrt{\frac{f_1^2}{f_2^2}}\right),$$

$$\xi_1 = k_1\left(y + \frac{\sqrt{\omega}t}{\frac{1}{k_1^2 - \frac{1}{4\omega}} - \epsilon}\right), \qquad \xi_2 = k_2\left(y + \frac{\sqrt{\omega}t}{\frac{1}{k_2^2 - \frac{1}{4\omega}} - \epsilon}\right), \qquad \omega > 0.$$

The corresponding solution represents a combination of a single (ϵ, ω)-soliton and a single (ϵ, ω)-cuspon.

In figure 5, we have depicted the solution profiles for seven different times, which describe the complete interaction process of one (ϵ, ω)-cuspon and one (ϵ, ω)-soliton. We can see that the (ϵ, ω)-cuspon and the (ϵ, ω)-soliton merge together at $t = 0$. As the time further evolves, the (ϵ, ω)-soliton separates from the (ϵ, ω)-cuspon little by little. Eventually, they regain their original forms except with some phase shifts.

At last, we point out that the head-on collision of one cuspon and one soliton has little study. Ferreira *et al* [16] and Dai and Li [27] have considered the interaction of a cuspon and a soliton whose peaks are on the opposite of the horizontal axis. Differently, this paper provides the head-on collision of a cuspon and a soliton whose peaks are always in the same side of the horizontal axis.

Acknowledgments

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